Computation of Normal Modes from Identified Complex Modes

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A technique is presented to compute a set of normal modes from a set of measured (damped) complex modes. The number of elements in the modal vectors, which is equal to the number of measurements, can be larger than the number of modes under consideration. It is also shown in this paper that the practice of normal mode approximation to complex modes can lead to considerably large errors when the modes are too complex. A numerical example and a simulated experiment are presented to illustrate the concepts discussed and to support the theory presented.

Nomenclature

$[C]$ = damping matrix
$[c]$ = modal damping matrix (diagonal)
f = frequency, Hz
I_j = imaginary part of the jth element of a complex modal vector
$[K]$ = stiffness matrix
$[k]$ = modal stiffness matrix (diagonal)
$[M]$ = mass matrix
$[m]$ = modal mass matrix (diagonal)
$n(t)$ = measurements noise
$\{p_j\}$ = jth eigenvector of the state variable equation
$\{Q_j\}$ = jth assumed modal vector
R_j = real part of the jth element of a complex modal vector
s_j = jth assumed characteristic root
$\{x(t)\}$ = state vector, $= \left\{ \begin{matrix} y(t) \\ \dot{y}(t) \end{matrix} \right\}$
$y(t)$ = free-response time function
α_1, α_2 = two angles defining sign (±) boundaries for the approximated normal mode elements
β_j = phase angle for the jth element of a normal modal vector (0.0 or 180.0 deg)
θ_j = phase angle of the jth element of a complex modal vector
ϕ_j = jth element of the normal modal vector
$\{\phi\}_j$ = jth normal modal vector
ψ_j = jth element of the complex modal vector
$\{\psi\}_j$ = jth complex modal vector
λ_j = jth characteristic root
ω = natural frequency, rad/s
ζ = damping factor
$[\ ]^T$ = transpose of a matrix
$[\ ]^{-1}$ = inverse of a matrix

Introduction and Background

MODAL vibration tests are carried out to experimentally determine a set of modal parameters for the structure under test. These modal parameters are usually used to verify, determine, or improve some analytical model of the structure.[1-8]

Most of the approaches that use experimentally determined modal parameters for dynamic modeling of structures use one or more of the following equations:

$$[M]^{-1}[K]\{\phi\} = \omega^2\{\phi\} \tag{1}$$

$$[\phi]^T[M][\phi] = [m] \tag{2}$$

$$[\phi]^T[K][\phi] = [k] \tag{3}$$

$$[\phi]^T[C][\phi] = [c] \tag{4}$$

In all these equations, the $\{\phi\}$ are the normal modes even though, in practice, the measured modes are the damped complex modes, which in some cases can be very different from normal modes. As a matter of fact, in vibration testing and analysis work it is frequently assumed that damping levels are very small and/or the damping matrix is proportional to either the mass or stiffness matrices, an assumption that is not valid for many of today's complex structures. Such assumptions and the lack of differentiation between normal and complex modes may be attributed to the lack of a tool to measure or compute the normal modes.

With the introduction of computer technology to modal identification in the early 1970s in both frequency domain[9,10] and time domain[11-18] techniques, the question of normal vs complex modes started to need answers. In frequency domain approaches, even with light damping and well-spaced modes, users frequently encountered a scatter of the phase angles associated with the measured modal vector.[14] Some researchers and users even went to the extent of questioning the test and data analysis procedures when the phase angles were not within ±10 deg at 0-180 deg.

It is to be noted also that measurement of phase angles in the frequency domain can be subject to high levels of errors especially in cases of high modal densities. This is due to the limited frequency resolution and the rapid change in the phase angles around the resonant frequencies. In some cases, the scatter of the phase angles of the modal vectors was due to the fact that the damping is nonproportional, and hence the mode shapes are complex. Time domain approaches to modal identification, which contain no assumptions regarding the level or proportionality of damping, also indicated that structures, in many cases, possess complex modes.

Normal Mode Approximation to Complex Modes

Normal modes are defined as modal vectors whose phase angles are either 0.0 or 180.0 deg. Such modes exist for extremely simple structures, that do not need any testing anyway. They also exist for structures with no damping or structures tailored with proportional damping, none of which represents today's complex structures.

Unlike normal modes, complex modes may possess any phase angle distribution. Each element of the modal vector is

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described by a real and imaginary part or an amplitude and phase angle relative to some arbitrary element. A scatter in the phase angles of as much as ±90.0 deg from 0.0 or 180.0 deg is not uncommon.

Recognizing the phase angle scatter for measured (complex) modes and the need for normal modes for use in equations such as (1-4), researchers and users have frequently used normal mode approximation to complex modes.

Figure 1a shows an element of a complex modal vector ψ_j, which is complex and can be expressed as,

$$\psi_j = R_j + iI_j \tag{5}$$

The approximate normal mode element ϕ_j corresponding to ψ_j is

$$\phi_j = \pm\sqrt{R_j^2 + I_j^2} \tag{6a}$$

where the assignment of a positive or negative sign, which is equivalent to 0.0 or 180.0 deg phase angle, depends on the angle θ_i ($\theta_i = \tan^{-1} I_j/R_j$) of the complex modal element and its relation to some arbitrary angles α_1 and α_2 as shown in Fig. 1b. In other words, the phase angle β_j for the approximated normal mode element ϕ_j is assigned according to the equations,

$$\beta_j = 0.0 \text{ deg} \qquad \alpha_2 < \theta_i < \alpha_1 \tag{6b}$$

$$\beta_j = 180.0 \text{ deg} \qquad \alpha_1 < \theta_i < \alpha_2 \tag{6c}$$

It is enough to state that, irrespective of the choice of α_1 and α_2, it is unacceptable to assign two different signs to two elements of the approximated normal modal vector because the phase angles of the corresponding elements of the complex modal vector differ by a fraction of a degree.

Such approximation can lead to erroneous and misleading results and conclusions. An example is the orthogonality check where the orthogonality of the measured modes with respect to the mass matrix is tested. Large off-diagonal terms may result not only because of errors in the mass matrix or inaccuracies in the identification process, but also because of the normal mode approximation to complex modes.

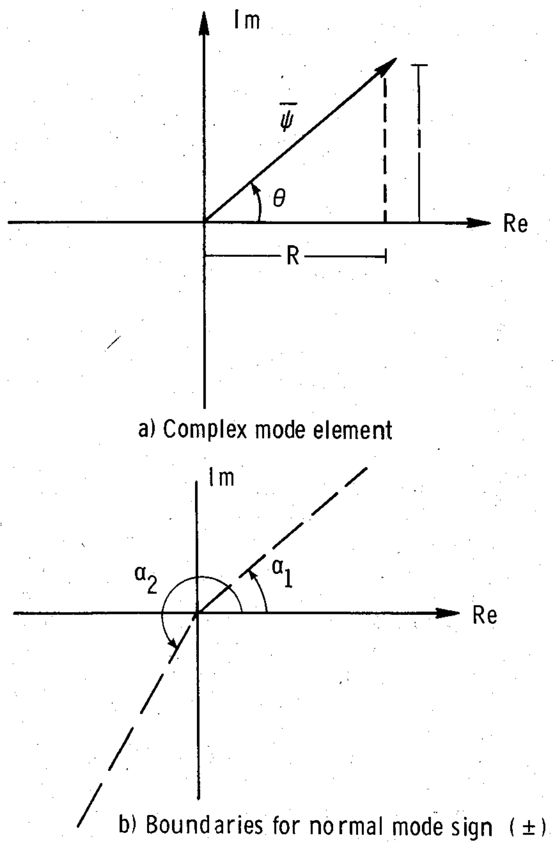

Fig. 1 Normal mode approximation to complex modes.

Numerical Example

The purpose of this example is to show that even though all the parameters used are exact:

1) Complex modes can be very different from normal modes, even for lightly damped modes and small non-proportionality in the damping matrix.

2) Large errors may result from assuming that normal modes approximated from complex modes are orthogonal with respect to the mass matrix.

The system used in this example is a 10 degree-of-freedom system. This system was constructed (simulated) by analytically generating 10 normal modes at 10 measurement stations of a simply supported beam, 10 undamped natural frequencies, and a stiffness matrix for the system. The natural frequencies were selected corresponding to 10.0, 12.0, 15.0, 20.0, 24.0, 30.0, 36.0, 43.0, 46.0, and 50.0 Hz. Then, a proportional damping matrix (equivalent to 1.0% modal damping factor for all 10 modes) and the mass matrix were computed from the assumed information.

To make the damping matrix nonproportional, the damping elements $C(3,3)$, $C(4,4)$, $C(3,4)$, and $C(4,3)$ were doubled. Complex modes, damping factors, and damped natural frequencies were computed for the system. Damping factors changed from 1.0% for all modes for proportional damping case to 2.6, 1.3, 1.2, 1.2, 1.1, 1.8, 2.9, 3.8, 1.7, and 1.0% for the nonproportional damping case. These damping factors are relatively small but nevertheless, some modes showed high levels of complexity. Table 1 shows the two most complex mode shapes, modes 9 and 10, listed with the corresponding normal modes. Phase angles of as much as 98.9 and 74.8 deg

Table 1 Comparison of complex and normal modes

9th mode			10th mode		
Normal $f = 46.00$ Hz ± Amplitude	Complex $\zeta = 1.75\%$ $f = 45.83$ Hz Amplitude	Phase	Normal $f = 50.00$ Hz ± Amplitude	Complex $\zeta = 1.04\%$ $f = 49.99$ Hz Amplitude	Phase
100.00	100.00	0.0	100.00	100.00	0.0
26.00	53.27	64.3	56.00	57.95	175.1
−136.00	144.46	−155.5	−0.00	9.94	−72.9
168.00	167.74	−0.3	56.00	54.85	7.3
−98.00	144.19	136.7	−100.00	102.48	176.0
−26.00	113.55	−98.9	114.00	119.61	−7.9
136.00	135.26	3.3	−98.00	102.24	172.3
−168.00	220.22	143.3	56.00	55.70	0.3
98.00	188.00	−54.8	−0.00	9.73	−74.8
26.00	36.38	48.7	−56.00	58.22	173.0

for modes 9 and 10 are noticed, respectively. Also, large differences in amplitudes exist between normal and complex modes.

To illustrate the large errors that may result from normal modes approximation to complex modes, approximated normal modes were used in checking their orthogonality with respect to the exact mass matrix. The orthogonality matrix results for different values of α_1 and α_2 are

$$\begin{bmatrix} 1.0000 & & & & & & & & & \\ 0.0003 & 1.0000 & & & & & & & & \\ -0.0000 & -0.0002 & 1.0000 & & & & & & & \\ -0.0004 & -0.0009 & -0.0003 & 1.0000 & & & & & & \\ 0.0010 & -0.0002 & 0.0027 & -0.0007 & 1.0000 & & & & & \\ 0.0010 & 0.0033 & 0.0097 & -0.0174 & 0.0013 & 1.0000 & & & & \\ 0.0003 & 0.0138 & 0.0054 & 0.0010 & 0.0029 & 0.0074 & 1.0000 & & & \\ 0.0013 & 0.0003 & -0.0005 & -0.0026 & -0.0071 & -0.0136 & 0.0070 & 1.0000 & & \\ -0.0597 & 0.0178 & 0.2339 & -0.0639 & -0.0895 & 0.0826 & -0.268 & 0.0316 & 1.0000 & \\ 0.0071 & -0.0005 & 0.0150 & -0.013 & -0.0091 & 0.0042 & -0.0209 & -0.0004 & -0.2281 & 1.0000 \end{bmatrix} \tag{7a}$$

$$\begin{bmatrix} 1.0000 & & & & & & & & & \\ 0.0003 & 1.0000 & & & & & & & & \\ -0.000 & -0.0002 & 1.0000 & & & & & & & \\ -0.0004 & -0.0009 & -0.0003 & 1.0000 & & & & & & \\ 0.0010 & -0.0002 & 0.0027 & -0.0007 & 1.0000 & & & & & \\ 0.0010 & 0.0033 & 0.0097 & -0.0174 & 0.0013 & 1.0000 & & & & \\ 0.0003 & 0.0138 & 0.0054 & 0.0010 & 0.0029 & 0.0074 & 1.0000 & & & \\ 0.0013 & 0.0003 & -0.0005 & -0.0026 & -0.0071 & -0.0136 & 0.0070 & 1.0000 & & \\ -0.2035 & 0.3549 & -0.1791 & 0.0311 & 0.0073 & -0.3108 & 0.1137 & 0.2265 & 1.0000 & \\ -0.0124 & -0.0144 & -0.1338 & 0.0016 & 0.0053 & -0.0241 & -0.0214 & 0.0254 & 0.1672 & 1.0000 \end{bmatrix} \tag{7b}$$

In Eq. (7a) α_1 and α_2 were chosen as 90 and 270 deg, while in Eq. (7b) they are 135 and 315 deg. Errors in the off-diagonal terms are as high as 23.29% for the first case and 35.49% for the second case.

Theory: Computation of Normal Modes

In this section, two approaches are presented to compute normal modes from a measured set of complex modes. The required data are a set of modal parameters such as may be identified from a modal survey test. These modal parameters are namely a set of complex modes $\{\psi\}_i$, $i=1,\ldots m$ and a set of corresponding characteristic roots λ_i, $i=1,\ldots m$ (and their complex conjugates). The modal vectors have n elements where $n>m$, which is a typical test situation. To compute the normal modes from this given set of complex modes, one of the following two approaches may be used.

Approach 1: Using an Oversized Mathematical Model

From the given modal parameters, displacement, velocity, and acceleration responses are formed according to the equations,

$$\{y(t)\} = \sum_{i=1}^{2m} \{\psi\}_i e^{\lambda_i t} + \{n_1(t)\} \tag{8a}$$

$$\{\dot{y}(t)\} = \sum_{i=1}^{2m} \lambda_i \{\psi\}_i e^{\lambda_i t} + \{n_2(t)\} \tag{8b}$$

$$\{\ddot{y}(t)\} = \sum_{i=1}^{2m} \lambda_i^2 \{\psi\}_i e^{\lambda_i t} + \{n_3(t)\} \tag{8c}$$

where $n_1(t)$, $n_2(t)$, $n_3(t)$ are added random noise of uniform distribution. These responses are then used in the state vector equation,

$$\begin{Bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{Bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{Bmatrix} y(t) \\ \dot{y}(t) \end{Bmatrix}$$

or

$$\{\dot{x}\} = [A]\{x\} \tag{9}$$

where $\{x\}$ is now the system's state vector containing the displacements and velocities responses. By repeating Eq. (9) for $2n$ time instants, the following equation is satisfied:

$$[\dot{X}] = [A][X] \tag{10}$$

where $[X]$ and $[\dot{X}]$ contain responses measured at the $2n$ time instants. From Eq. (10) the $[A]$ matrix can be identified as,

$$[A] = [\dot{X}][X]^{-1} \tag{11}$$

By computing matrix $[A]$, the $[M^{-1}K]$ matrix gives normal modes according to the eigenvalue equation,

$$[M^{-1}K]\{\phi\}=\omega^2\{\phi\} \qquad (12)$$

Naturally, without any noise, the matrix $[X]$ is singular since the number of degrees of freedom is larger than the number of modes present in the responses. A small amount of noise makes the inversion of $[X]$ possible for the purpose of extracting modal information. For example, noise to signal ratios of as little as 0.000001 were used[17] to invert a 600×600 matrix of a rank of 4 without signs of ill-conditioning on a 60 bit word computer. Higher levels of noise may be needed for computers with less accuracy.

The mechanism on which the success of this approach is based can be explained as follows.

The state vector's $2n$ free-response functions containing modal information from m structural modes of vibration can be expressed as,

$$\{x(t)\}=\sum_{k=1}^{2m}\{p\}_k e^{\lambda_k t} \qquad (13)$$

where $\{p\}$ represents the m complex pairs of the systems independent eigenvectors. If noise-free responses are used in the identification algorithm, the math model must have exactly m degrees of freedom for unique identification. If more than m degrees of freedom are allowed, the $[X]$ matrix is singular.

In experimental work, however, measured responses always contain a certain amount of noise (or as in this case a small amount of noise is added on purpose). These noisy responses can be expressed as,

$$\{x(t)\}=\sum_{k=1}^{2m}\{p\}_k e^{\lambda_k t}+\{n(t)\} \qquad (14)$$

In previous applications[11-17] it was found that using noisy responses in the identification process, with the number of degrees of freedom larger than m, yielded good results without encountering singularity. The results even improved as the math model size was increased. The qualitative explanation for this situation is that the extra degrees of freedom act as outlets for the noise. In this case, the noisy responses can be expressed as,

$$\{x(t)\}=\sum_{k=1}^{2m}\{p\}_k e^{\lambda_k t}+\sum_{k=2m+1}^{2n}\{N\}_k e^{r_k t} \qquad (15)$$

in which the noise is modeled as a combination of $(2n-2m)$ complex exponential functions. Since the value of m, the number of excited modes, is a characteristic of the structural response and not the data analysis process, additional exponential functions are allowed to represent the noise in the math model as n is increased. This results in a higher-order fit for the noise portion of the responses, reducing residuals that would otherwise be included in the signal portion of the responses.

Approach 2: Using Assumed Modes

The given set of complex modal parameters satisfy the equation

$$[M^{-1}K \quad M^{-1}C]\begin{Bmatrix}\psi_i\\ \lambda_i\psi_i\end{Bmatrix}=\{-\lambda_i^2\psi_i\} \quad (i=1,\ldots m) \qquad (16)$$

Since we have only m modes and the system has n degrees of freedom, Eq. (16) cannot be solved for $[M^{-1}K\,M^{-1}C]$. Let us assume that there exists a set of vectors $\{Q\}_j$ and a set of characteristic roots s_j, $j=m+1,\ m+2,\ldots n$. This set of assumed parameters are selected such that

$$\lambda_i \neq s_j \qquad (17a)$$

$$\{Q\}_j \neq [\psi_1\ \ \psi_2\ldots\psi_m]\{a\} \qquad (17b)$$

where $\{a\}$ is any vector of coefficients. Equation (17b) implies that $\{Q\}_j$ and $\{\psi\}_i$ for all i and j form a linearly independent set of vectors. In such a case, it can be written that

$$[M^{-1}K \quad M^{-1}C]\begin{Bmatrix}Q_j\\ s_jQ_j\end{Bmatrix}=\{-s_j^2Q_j\} \quad (j=m+1,\ m+2,\ldots n) \qquad (18)$$

and Eqs. (16) and (18) can be solved for $[M^{-1}K\,M^{-1}C]$ from which normal modes are computed according to Eq. (12).

To illustrate the soundness of this second approach, let it be assumed that there exists a hypothetical system whose n free response time functions are linear combinations of the two independent sets of modes $\{\psi\}$ and $\{Q\}$. These responses can then be expressed as

$$\{y(t)\}=\sum_{k=1}^{2m}\{\psi\}_k e^{\lambda_k t}+\sum_{j=1}^{2n-2m}\{Q\}_j e^{s_j t}$$

$$=[\psi_1\psi_2\ldots\psi_{2m}]\begin{Bmatrix}e^{\lambda_1 t}\\ e^{\lambda_2 t}\\ \ldots\\ e^{\lambda_{2m} t}\end{Bmatrix}+[Q_1Q_2\ldots Q_{2n-2m}]\begin{Bmatrix}e^{s_1 t}\\ e^{s_2 t}\\ \ldots\\ e^{\lambda_{2n-2m} t}\end{Bmatrix}$$

$$=[\psi_1\psi_2\ldots\psi_{2m}Q_1Q_2\ldots Q_{2n-2m}]\begin{Bmatrix}e^{\lambda_1 t}\\ e^{\lambda_2 t}\\ \ldots\\ e^{\lambda_{2m} t}\\ e^{s_1 t}\\ e^{s_2 t}\\ \ldots\\ e^{s_{2n-2m} t}\end{Bmatrix} \qquad (19)$$

The responses of Eq. (19) are typical of a second order dynamic system whose state vector equation is

$$\begin{Bmatrix}\dot{y}\\ \ddot{y}\end{Bmatrix}=\begin{bmatrix}0 & I\\ A & B\end{bmatrix}\begin{Bmatrix}y\\ \dot{y}\end{Bmatrix}$$

where the $[A]$ matrix represents the inertia-stiffness information and the $[B]$ matrix represents the inertia-damping characteristics.

If these responses, as expressed in Eq. (19), are to be used in any identification algorithm, the vectors $\{\psi\}$ and $\{Q\}$ and the characteristic roots λ and s will be uniquely identified. The identified properties of the initial set of modes $\{\psi\}$ should be unique and independent of the assumed Q and s as long as the conditions of Eq. (17) are satisfied.

An appropriate selection for the set of assumed modal parameters would be from the structure's finite element model. Higher analytical modes, other than the measured ones, are highly recommended for such a use.

It is extremely important to point out that $[M^{-1}K]$ and $[M^{-1}C]$ obtained from either approach are not unique since

Table 2 Identified complex mode and normal mode

Mode 9						Mode 10					
Theoretical			Identified			Theoretical			Identified		
Normal $f=46.00$ Hz	Complex $\zeta=1.75\%$ $f=45.83$ Hz		Normal $f=46.00$ Hz	Complex $\zeta=2.60\%$ $f=45.93$ Hz		Normal $f=50.00$ Hz	Complex $\zeta=1.04\%$ $f=50.00$ Hz		Normal $f=50.00$ Hz	Complex $\zeta=1.74\%$ $f=50.00$ Hz	
± Amplitude	Amplitude	Phase	± Amplitude	Amplitude	Phase	± Amplitude	Amplitude	Phase	± Amplitude	Amplitude	Phase
100.00	100.00	0.0	100.00	100.00	0.0	100.00	100.00	0.00	100.00	100.00	0.0
26.00	53.27	64.3	25.18	64.93	59.0	−56.00	57.95	175.1	−55.07	57.27	170.7
−136.00	144.46	−155.5	−136.76	152.63	−154.0	0.00	9.94	−72.9	0.39	13.54	−74.4
168.00	167.74	−0.3	170.00	164.89	−2.2	56.00	54.85	7.3	54.08	53.19	8.8
−98.00	144.19	136.7	−97.65	135.86	132.6	−100.00	102.48	176.0	−94.95	97.74	175.6
−26.00	113.55	−98.9	−36.95	107.75	−109.2	114.00	119.61	−7.9	109.92	116.18	−9.6
136.00	135.26	3.3	145.63	140.38	−0.3	−98.00	102.24	172.3	−90.82	94.78	171.5
−168.00	220.22	143.3	−167.51	202.69	143.7	56.00	55.70	0.3	59.42	58.61	−1.1
98.00	188.00	−54.8	94.81	164.94	−54.7	−0.00	9.73	−74.8	−0.44	4.01	−48.1
26.00	36.38	48.7	28.32	36.51	41.5	−56.00	58.22	173.0	−55.36	57.71	170.9

they are functions of the introduced noise or the assumed modes. However the set of normal modes, corresponding to the set of given complex modes, was found to be independent of the introduced small levels of noise or the assumed modes.

Simulated Experiment

To test the validity of the theories presented in this paper, the 10 degrees-of-freedom system previously discussed in the section on numerical example is used here as a simulated test structure. Response time histories containing contribution from the last four modes measured at the 10 stations were generated. The last four modes were selected because the last two modes show a high level of complexity. Simulated measurements noise was added to these responses, with a noise/signal rms ratio of 20%, to represent conditions in a real vibration test. From these responses, the complex modes and characteristic roots were identified, using the time domain approach.[11] Normal modes were then computed using the two approaches presented here. A noise to signal ratio of 0.00001 was used for Eqs. (8). The assumed modes approach produced results identical to those of the oversized math model approach.

Table 2 lists the identified complex modes and the computed normal modes for the last two modes. A close examination of the computed normal modes, in comparison with the theoretical ones, indicate the validity of the approaches presented.

Using the identified complex modes and the computed normal modes, the orthogonality check matrices are

$$\begin{bmatrix} 1.0000 & & & \\ -0.0438 & 1.0000 & & \\ -0.0018 & -0.0300 & 1.0000 & \\ -0.0205 & 0.0365 & -0.2091 & 1.0000 \end{bmatrix} \tag{20a}$$

$$\begin{bmatrix} 1.0000 & & & \\ -0.0438 & 1.0000 & & \\ 0.0899 & 0.0999 & 1.0000 & \\ -0.0282 & 0.0626 & -0.4802 & 1.0000 \end{bmatrix} \tag{20b}$$

$$\begin{bmatrix} 1.0000 & & & \\ -0.0464 & 1.0000 & & \\ 0.0228 & -0.0353 & 1.0000 & \\ -0.0082 & 0.0531 & -0.0546 & 1.0000 \end{bmatrix} \tag{20c}$$

In Eqs. (20a) and (20b) approximated normal modes were used with (90, 270 deg) and (135, 315 deg) for (α_1, α_2), respectively. In Eq. (20c) the computed normal modes were used. Errors of 21 and 48% are noticed in the off-diagonal terms for cases a and b, respectively, while the maximum error for case c was only 5%.

Conclusions

It is shown in this paper that even for low levels of damping for structures with nonproportional damping, complex modes can be very different from normal modes. In such cases, normal mode approximation to complex modes may lead to large errors in mass-weighted orthogonality checks or in any other use of these complex modes approximated as normal modes.

A technique is presented to compute normal modes from measured complex modes. Computed normal modes eliminate possible errors that may result from using normal mode approximation to complex modes produced by nonproportional damping.

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